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THE PRESSURE FIELD OF A UNIFORMLY LOADED RECTANGULAR WING IN A SUPERSONIC FLOW

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Two problems are distinguished in wing aerodynamics: the direct aerodynamic problem (determination of the field of gas-dynamic flow parameters from a specified wing geometry) and the inverse aerodynamic problem (determination of the field of flow parameters from a specified load on the wing). For a thin finite-span wing in supersonic flow [1–3], the solution of the direct problem in a linear formulation is represented as the potential

$$\Phi_1(x,y,z) = -\frac{1}{\pi}\iint_s \frac{\Phi'_\eta(\xi,\zeta)}{\sqrt{(x-\xi)^2-[(z-\zeta)^2+y^2]}}\,ds \tag{1}$$

and the solution of the inverse problem is represented as the potential

$$\Phi(x,y,z) = \frac{y}{\pi}\iint_s \frac{\Phi'_\xi(\xi,\zeta)(x-\xi)}{[(z-\zeta)^2+y^2]\sqrt{(x-\xi)^2-[(z-\zeta)^2+y^2]}}\,ds. \tag{2}$$

Here s is the region of dependence of the point $P(x,y,z)$ in the base plane $\eta = 0$; $\Phi'_\eta(\xi,\zeta)$ and $\Phi'_\xi(\xi,\zeta)$ are the flow downwash and the pressure difference in the plane $\eta = 0$, respectively; (x,y,z) are the coordinates of the point P in the rectangular Cartesian coordinate system related to the physical rectangular coordinate system by $x_1 = \sqrt{\mathrm{M}^2-1}\,x$, $y_1 = y$, and $z_1 = z$. The region of dependence s is bounded by the line L of intersection of the characteristic cone whose vertex is at point P with the plane $\eta = 0$:

$$(x-\xi) - \sqrt{(z-\zeta)^2+y^2} = 0. \tag{3}$$

Many papers are devoted to solving direct aerodynamic problems following (1), with a simple kernel. The solution of problems in the form of potential (2) involves difficulties related to the complex structure of the integrand kernel and to the substantial singularity when the point P tends to the base plane ($y \to 0$) [4].[1] When the gas-dynamic flow parameters (derivatives of the velocity potential) are found, the power of the integrand singularities increases, sometimes making impossible formal differentiation. In some cases, differentiation gives rise to singularities that make the integrals divergent. The condition of existence of integrals in the sense of Hadamard is often used [2]. The introduction of such symbols not only complicates the realization of the algorithms of solution but, sometimes, also requires justification of physically absurd results. However, a careful account of the singularities of the kernels of integral operators and some smoothness conditions imposed on the governing parameters of the problem allow one to represent gas-dynamic flow parameters using bounded functions [3, 4].

In this paper, the pressure field (the most illustrative gas-dynamic flow characteristic) of a uniformly loaded wing with a rectangular planform is represented in terms of elementary functions. This result, being of interest by itself (the pressure field induced by some rectangular element of aircraft), can be used in realizing the algorithm of solution of inverse problems for complex-shaped wings when the wing projection onto the base plane is divided into rectangular cells, and also as a test for numerical calculations. Since the differentiation and integration operations required to solve the problem are too complicated and cumbersome, they are omitted.

[1]The notation of the coordinate axes y, η and z, ζ in the figure in [4] should be interchanged.

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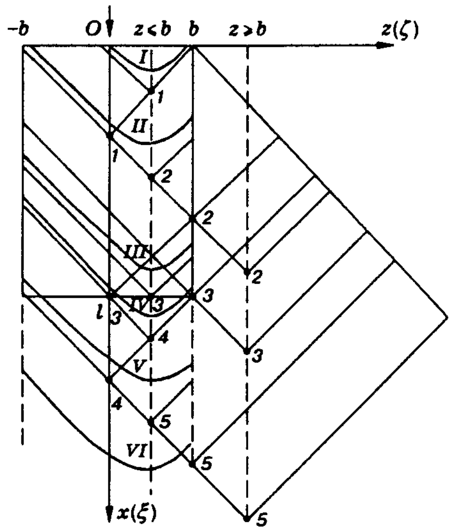

Fig. 1

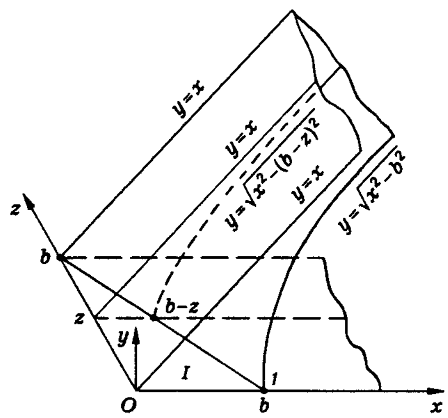

Fig. 2

Let us consider the supersonic flow around a rectangular wing with span $2b$ and length l ($l \geqslant 2b$). Figure 1 shows the traces of the cones of dependence of points $P(x,y,z)$ in the plane $\eta = 0$. Characteristic regions with typical forms of solution can be distinguished.

For points in the plane $z = \text{const}$, $0 < z < b$, there are six typical regions of solution.

Region I includes points $P(x,y,z)$; the trace L of their cone [hyperbola (3)] intersects twice the leading edge of the wing $\xi = 0$, without crossing the side edges.

In region II, the hyperbola L intersects the leading edge $\xi = 0$ and the side edge $\zeta = b$.

In region III, the hyperbola L intersects the side edges $\zeta = -b$ and $\zeta = b$ without crossing the leading and trailing edges.

In region IV, the hyperbola L intersects twice the trailing edge $\xi = l$ and the side edges $\zeta = -b$ and $\zeta = b$.

In region V, the hyperbola L crosses one time the trailing edge $\xi = l$ and the side edge $\zeta = -b$.

In region VI, the hyperbola L does not cross the wing projection; this region is extended infinitely downstream.

Points 1–5 (Fig. 1) in the plane $z = \text{const}$ $(0 < z < b)$ are the boundaries (downstream) for points $P(x, y = 0, z)$ in regions I–V. For points $P(x, y = 0, z)$, the hyperbola L is degenerated into the characteristic curves emanating from points 1–5.

For points $P(x,y,z)$ lying in the plane of symmetry $z = 0$, there are no regions II and V. For points $P(x,y,z)$ lying in the plane of the side edge $z = b$, region I degenerates into the curve $y = x$ (into the point $x = 0$, $y = 0$ in the plane $\eta = 0$); there is no region IV. For points $P(x,y,z)$ in the plane $z = \text{const}$, $z > b$, regions II, III, V, and VI exist.

Figure 2 shows isometrically region I located above the right half of the wing. It is bounded by the leading characteristic plane $y = x$ and by the surface $y = \sqrt{x^2 - (b-z)^2}$. In the plane $z = b$, region I degenerates into the characteristic curve $y = x$. At large distances from the wing ($x \geqslant y \geqslant H \gg l \geqslant b$), the surface $y = \sqrt{x^2 - (b-z)^2}$ approaches asymptotically the leading characteristic plane $y = x$.

Figure 3 shows a sketch of sections of regions I–VI by the symmetry plane $z = 0$. The characteristic $y = x$ separates the undisturbed flow region and region I. The curve $y = \sqrt{x^2 - b^2}$ is the boundary between regions I and III. The characteristic $y = x - l$ is the boundary between regions III and IV. The curve $y = \sqrt{(x-l)^2 - b^2}$ separates regions IV and VI.

After cumbersome integration and differentiation operations, the pressure field for a uniformly loaded

$[\Phi'_\xi(\xi,\zeta) = p_0]$ rectangular wing is expressed in terms of elementary functions. Introducing the function

$$\arcsin\left\{1 - 2y^2 \frac{(x-\xi)^2 - [(z-\zeta)^2 + y^2]}{[(x-\xi)^2 - y^2][(z-\zeta)^2 + y^2]}\right\} = \operatorname{arc}(\xi,\zeta), \tag{4}$$

we write the resulting values of $\Phi'_x(x,y,z)$ in regions I–VI.

In the plane $z = \text{const}$, $0 \leqslant z \leqslant b$, the solution is

$$\Phi'_x = p_0, \qquad y \leqslant x \leqslant \sqrt{(z-b)^2 + y^2} \qquad \text{(region I)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[\frac{3}{2}\pi + \operatorname{arc}(0,b)\Big], \quad \sqrt{(z-b)^2+y^2} \leqslant x \leqslant \sqrt{(z+b)^2+y^2} \qquad \text{(region II)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[\pi + \operatorname{arc}(0,b) + \operatorname{arc}(0,-b)\Big], \quad \sqrt{(z+b)^2+y^2} \leqslant x \leqslant (l+y) \qquad \text{(region III)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[-\pi + \operatorname{arc}(0,b) + \operatorname{arc}(0,-b)\Big], \quad (l+y) \leqslant x \leqslant l + \sqrt{(z-b)^2+y^2} \quad \text{(region IV)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[-\frac{\pi}{2} + \operatorname{arc}(0,b) + \operatorname{arc}(0,-b) - \operatorname{arc}(l,b)\Big], \quad l + \sqrt{(z-b)^2+y^2} \leqslant x \leqslant l + \sqrt{(z+b)^2+y^2} \quad \text{(region V)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[\operatorname{arc}(0,b) + \operatorname{arc}(0,-b) - \operatorname{arc}(l,b) - \operatorname{arc}(l,-b)\Big], \quad x \geqslant l + \sqrt{(z+b)^2+y^2} \quad \text{(region VI)}.$$

In the plane $z = \text{const}$, $z \geqslant b$, the solution is

$$\Phi'_x = \frac{p_0}{2\pi}\Big[\frac{\pi}{2} - \operatorname{arc}(0,b)\Big], \quad \sqrt{(z-b)^2+y^2} \leqslant x \leqslant \sqrt{(z+b)^2+y^2} \qquad \text{(region II)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[\operatorname{arc}(0,-b) - \operatorname{arc}(0,b)\Big], \quad \sqrt{(z+b)^2+y^2} \leqslant x \leqslant l + \sqrt{(z-b)^2+y^2} \quad \text{(region III)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[-\frac{\pi}{2} + \operatorname{arc}(0,-b) - \operatorname{arc}(0,b) + \operatorname{arc}(l,b)\Big], \quad l + \sqrt{(z-b)^2+y^2} \leqslant x \leqslant l + \sqrt{(z+b)^2+y^2} \quad \text{(region V)},$$

$$\Phi'_x = \frac{p_0}{2\pi}\Big[\operatorname{arc}(l,b) - \operatorname{arc}(l,-b) - \operatorname{arc}(0,b) + \operatorname{arc}(0,-b)\Big], \quad x \geqslant l + \sqrt{(z+b)^2+y^2} \quad \text{(region VI)}.$$

The boundaries of the regions of solution are surfaces in the disturbed flow region.

In the above solutions, $\operatorname{arc}(0,b)$, $\operatorname{arc}(0,-b)$, $\operatorname{arc}(l,b)$, and $\operatorname{arc}(l,-b)$ are the values of function (4) at points $(\xi = 0, \zeta = b)$, $(\xi = 0, \zeta = -b)$, $(\xi = l, \zeta = b)$, and $(\xi = l, \zeta = -b)$, respectively. For points $P(x, y = 0, z)$ lying in the base plane, $\operatorname{arc}(0,\pm b) = \operatorname{arc}(l,\pm b) = \pi/2$. For points $P(x,y,z = 0)$ in the symmetry plane of the wing, $\operatorname{arc}(0,b) = \operatorname{arc}(0,-b)$ and $\operatorname{arc}(l,b) = \operatorname{arc}(l,-b)$. For points $P(x,y,z = b)$ in the plane of the right-hand side edge, $\operatorname{arc}(0,b) = \operatorname{arc}(l,b) = -\pi/2$. Furthermore, the following equalities are valid at the boundaries of regions I–VI: $\operatorname{arc}(0,b) = \pi/2$ for regions I and II [on the surface $x = \sqrt{(z-b)^2+y^2}$], $\operatorname{arc}(0,-b) = \pi/2$ for regions II and III [on the surface $x = \sqrt{(z+b)^2+y^2}$], $\operatorname{arc}(l,b) = \pi/2$ for regions IV and V [on the surface $x = l + \sqrt{(z-b)^2+y^2}$], and $\operatorname{arc}(l,-b) = \pi/2$ for regions V and VI [on the surface $x = l + \sqrt{(z+b)^2+y^2}$].

Taking into account the above-mentioned properties of function (4), one can easily analyze the pressure field obtained from the solutions in regions I–VI.

In the base plane $y = 0$, the pressure difference is $\Phi'_x = 0$ everywhere outside the wing projection S. As should be expected, $\Phi'_x = p_0$ on the projection S.

In the plane $z = b$, the solution is not discontinuous. The solution for $z \leqslant b$ is continuously transformed into the solution for $z \geqslant b$; the solutions are glued in regions II, III, V, and VI.

For $|z| \leqslant b$, the solution is continuous at the boundaries of regions I and II, II and III, IV and V, and V and VI. The solution is discontinuous at the boundary of regions III and IV.

For $z \geqslant b$, the solution in one region is continuously transformed into the solution of the next region (taking into account that, for $z = b$, regions I and VI degenerate into the characteristic curves $y = x$ and $y = x - l$, respectively).

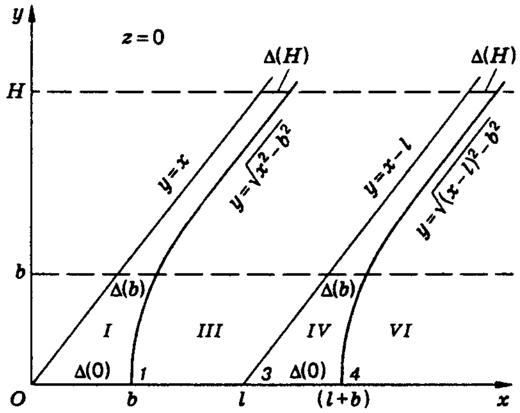

Fig. 3

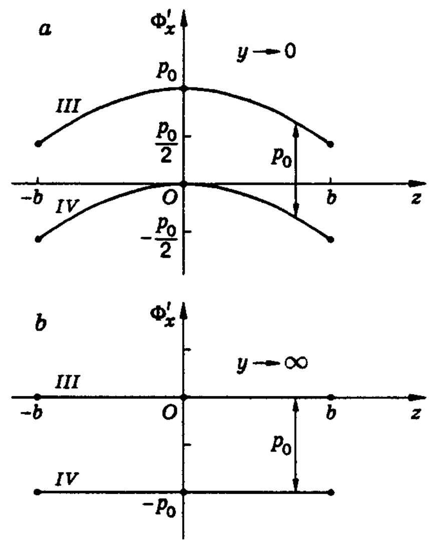

Fig. 4

The boundary of regions III and IV is the characteristic plane $y = x - l$ passing through the trailing edge of the wing $\xi = l$. The pressure difference is constant, $[\Phi'_x(\mathrm{III}) - \Phi'_x(\mathrm{IV})] = p_0$, over the entire plane $y = x - l$. However, unlike the constant pressure difference p_0 in the leading characteristic plane, which is the boundary between the undisturbed flow and region I, where the pressure equals zero upstream of the discontinuity surface and is constant and equal to p_0 behind this surface, the pressure on the "rear" characteristic surface is a variable quantity on both sides of it.

Figure 4 shows the spanwise variation of Φ'_x in the characteristic plane $y = x - l$ in regions III (curves III) and IV (curves IV). Near the base plane ($y \to 0$, Fig. 4a), Φ'_x changes from p_0 in the symmetry plane $z = 0$ to $p_0/2$ in the plane of the wing side edge in region III, and from zero to $-p_0/2$ in region IV. The pressure difference is the same and equal to p_0 everywhere. At a large distance from the base plane ($y \to \infty$, Fig. 4b), $\Phi'_x = 0$ in region III and $\Phi'_x = -p_0$ in region IV with second-order accuracy. The pressure difference is again equal to p_0 along the entire wing span.

Let us analyze the Φ'_x distribution in the symmetry plane of the wing $z = 0$. The flow pattern in this plane is presented in Fig. 3. Let us distinguish several cross sections $y = y^* = \text{const}$: $O, b, H \gg l \geqslant b, \infty$. The length of regions I and IV in these sections is $\Delta(y) = \sqrt{b^2 + y^2} - y$. For the chosen sections $\Delta(0) = b$, $\Delta(b) = (\sqrt{2} - 1)b$, $\Delta(H) = b^2/2H$ (with second-order accuracy), and $\Delta(\infty) = 0$ (regions I and IV asymptotically degenerate into the characteristic curves $y = x$ and $y = x - l$, respectively). In the cross section $y^* = 0$ (Fig. 5a), as follows from the formulation of the problem, $\Phi'_x = p_0$ on the wing ($\tau \leqslant l$) and $\Phi'_x = 0$ in the wake behind the wing ($\tau > l$).

There is a new variable $\tau = x - y^*$ in Fig. 5. In the new variables in all cross sections, $\tau = 0$ corresponds to the leading characteristic curve $y = x$ and $\tau = l$ to the characteristic curve $y = x - l$. As can be seen from Fig. 5a–d, the quantity Φ'_x undergoes a jump $[\Phi'_x(\mathrm{III}) - \Phi'_x(\mathrm{IV})] = p_0$ in the characteristic curve $y = x - l$ in all cross sections $y = y^*$, as was noted above in the analysis of Fig. 4. In the cross section $y^* = b$ (Fig. 5b), the length of the constant-pressure region decreases to the value $(\sqrt{2} - 1)b$, after which Φ'_x decreases continuously to the value of $0.55p_0$ on the characteristic curve $y = x - l$, where it undergoes the jump p_0. After the jump, $\Phi'_x < 0$, decreasing in absolute value, tends asymptotically to zero as $\tau \to \infty$. At large distances from the wing ($y^* = H \gg l \geqslant b$, Fig. 5c), the disturbances from the wing S with second-order accuracy are concentrated in narrow regions I and IV with lengths $\Delta(H)$, where $\Phi'_x = p_0$. The results presented in Fig. 5c can be treated as sonic-boom characteristics in the symmetry plane of a rectangular wing with span $2b$: the leading pressure jump of intensity p_0 and the trailing rarefaction jump of intensity $-p_0$, the length of the jumps being $\Delta(H) = b^2/2H$. At very large distances ($y \to \infty$, Fig. 5d), the length $\Delta(\infty) \to 0$, and regions I and IV

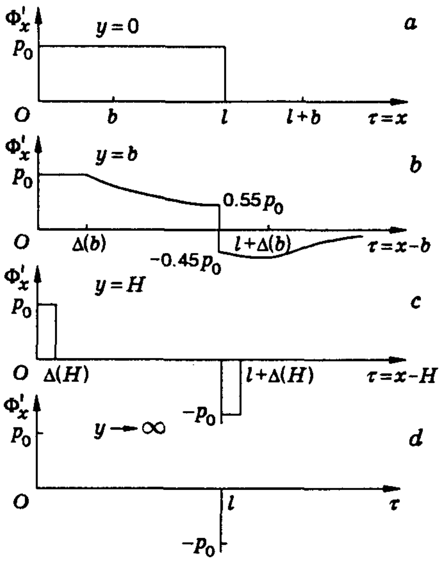

Fig. 5

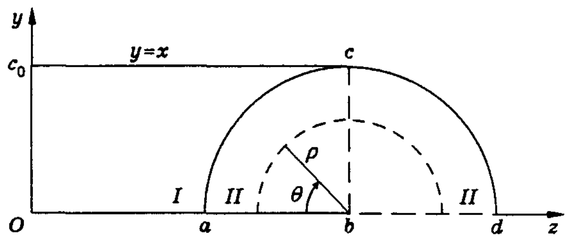

Fig. 6

degenerate into the curves $y = x$ and $y = x - l$, where Φ'_x has a discontinuity p_0.

Let us analyze the Φ'_x distribution in the plane x = const passing through regions I and II (the edge effect region). In region II, the solution is written as

$$\Phi'_x = \frac{p_0}{2\pi}\left[\frac{3}{2}\pi + \operatorname{arc}(0,b)\right] \quad \text{for } z \leqslant b; \tag{5}$$

$$\Phi'_x = \frac{p_0}{2\pi}\left[\frac{\pi}{2} - \operatorname{arc}(0,b)\right] \quad \text{for } z \geqslant b. \tag{6}$$

Figure 6 shows the section plane x = const of the disturbed region. The curve Ob is the trace of the right half of the rectangular wing, and the curve c_0cd is the trace of the leading characteristic surface (c_0c is the trace of the characteristic plane $y = x$). The trace of region I is the rectangle Oc_0caO, and that of region II is a semicircle $acdba$ of radius x.

Let us write the expression $\operatorname{arc}(0, b)$ in extended form

$$\operatorname{arc}(0,b) = \arcsin\left\{1 - 2y^2\,\frac{x^2 - [(z-b)^2 + y^2]}{(x^2 - y^2)[(z-b)^2 + y^2]}\right\}.$$

We substitute the variables $y = \rho\sin\theta$ and $(b - z) = \rho\cos\theta$; the direction of counting of the angle θ is shown in Fig. 6. According to the substitution of variables, $[(z - b)^2 + y^2] = \rho^2$, and, hence,

$$\operatorname{arc}(0,b) = \arcsin\left\{1 - 2\sin^2\theta\,\frac{x^2 - \rho^2}{x^2 - \rho^2\sin^2\theta}\right\}. \tag{7}$$

Solution for $z \leqslant b$ [equalities (5) and (7)]. At the interface between regions I and II (on the circumference ac when $\rho = x$), we have $\Phi'_x = p_0$; the solution in region I is continuously transformed into the solution in region II (this result was noted above in the general flow analysis). In the base plane $y = 0$ (on the segment ab of the wing trace when $\theta = 0$), $\Phi'_x = p_0$, which is consistent with the conditions imposed by the formulation of the problem. In the side edge plane $z = b$ (when $\theta = \pi/2$), $\Phi'_x = p_0/2$.

Solution for $z \geqslant b$ [equalities (6) and (7)]. In the side edge plane $z = b$ (when $\theta = \pi/2$), the solution is continuous (this result was noted above). For points $z \geqslant b$ on the slot of the base plane (on the segment bd when $\theta = \pi$), we have $\Phi'_x = 0$, which is also in agreement with the formulation of the problem. For points on the diffraction portion of the leading characteristic surface (on the circumference cd when $\rho = x$), we have $\Phi'_x = 0$, which is consistent with the generally accepted boundary conditions in a linear formulation.

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